

Hypocoercivity and the Cosmic Gas:

A study of unobserved law and Cédric Villani's Methods in Astrophysical Kinetics

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Abstract

We study unobserved law of nature and present a comprehensive review of the mathematical theories of kinetic equations developed by Cédric Villani and collaborators, with a specific focus on their under-utilized potential in theoretical astrophysics and cosmology. Our abstract will be imperfect, the topic of gas is a difficult and poorly understudied topic but we leave room for future corrections. The idea is to ground this inquiry in physical reality: consider the trajectory of a single photon γ with wave 4-vector $k^\mu = (\omega/c, \mathbf{k})$ traversing a Cosmic Gas cloud. As it pierces the halo, it does not encounter a static void, but a dynamic, seething plenum of phase-space filaments. The photon's frequency shift $\frac{\Delta\nu}{\nu} = \int \hat{n} \cdot \nabla \Phi dl$ encodes the "breathing" of the potential well—a direct probe of the Landau damping mechanism at work. This interaction reveals the fundamental reality of the gas cloud not merely as matter, but as a standing wave of *energy*, where the virial theorem $2K + W = 0$ is constantly tested by non-linear fluctuations. However, this standard picture relies on the implicit validity of Maxwell's equations in the continuous limit. We venture a bold hypothesis: that Maxwell's formalism is incomplete when describing the turbulent energy exchanges in the cosmic gas, requiring a *photonic extension* that treats the electromagnetic field not as a smooth manifold but as a granular, stochastic medium. We argue that what we perceive as "dark matter" phenomena may essentially be the specific signature of this "Maxwellian Incompleteness" on galactic scales. The "Cosmic Gas"—defined here as the coupled system of self-gravitating Dark Matter, the baryonic Intergalactic Medium (IGM), and emergent scalar fields—is governed by the Vlasov-Poisson-Boltzmann system, a set of non-linear partial differential equations whose rigorous properties have been elucidated only recently. We map Villani's Fields Medal-winning work on Landau Damping to the stability of Dark Matter halos, arguing that the Mouhot-Villani Theorem provides the rigorous justification for the stability of collisionless virialized structures. Furthermore, we explore the theory of *Hypocoercivity*, which describes convergence to equilibrium in systems with degenerate dissipation (where noise acts only on velocities but confinement is spatial). We propose that Hypocoercivity offers a superior framework for understanding the thermalization of the Warm-Hot Intergalactic Medium (WHIM) and galaxy clusters compared to standard collisional relaxation times. Finally, we discuss Optimal Transport theory, specifically

the concept of displacement convexity, as the geometric dual to the Zeldovich approximation in Large Scale Structure formation. The culmination of our abstract will feature the innovative maxwell extension.

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1 Introduction

1.1 The “Cosmic Gas” Paradigm

Contemporary cosmology often treats the constituents of the universe as distinct, uncoupled fluids during linear evolution: Dark Matter is modeled as a collisionless pressure-free fluid, baryons as an ideal gas, and photons as a radiation background. While effective for predicting the Cosmic Microwave Background (CMB) spectrum, this separation breaks down in the non-linear regime of structure formation. To understand galaxy formation, feedback, and the thermal history of the universe, one must consider the *Cosmic Gas* in its totality: a unified, multi-phase, self-gravitating medium defined on a high-dimensional phase space.

Consider, for a moment, the microscopic journey of a single photon carrier particle, γ , originating from a background quasar and entering the virial radius of a massive galaxy cluster. Its Lagrangian path $x^\mu(\lambda)$ is determined by the null geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \quad (1)$$

where the Christoffel symbols $\Gamma_{\alpha\beta}^\mu$ are perturbed by the fluctuating gravitational potential $\delta\Phi(\mathbf{x}, t)$ of the gas itself. Here lies the crux of the Maxwellian inadequacy. The classical field tensor $F_{\mu\nu}$ assumes a smooth differentiable manifold, yet the photon traversing the gas probes the *discreteness* of the phase-space filaments. We posit that Maxwell's equations must be extended to include a 'stochastic potential' term Ξ^ν , such that the modified Ampère-Maxwell law reads:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu + \underbrace{\Xi_{stoch}^\nu(\Phi, f)}_{\text{Granular Fluctuations}} \quad (2)$$

where Ξ_{stoch}^ν represents the non-linear coupling between the photon field and the granular fluctuations of the gas energy density $\delta T_{\mu\nu}$. This term, usually averaged to zero in mean-field theory, becomes dominant in the "seething" regime of violent relaxation. The extensive mathematical consequences of this term, and its relation to Villani's damping, are the subject of our inquiry. As this photon traverses the Intergalactic Medium (IGM), it does not merely pass through "dust"; it interacts with a collisionless plasma that is in a constant state of "violent relaxation." The electron density $n_e(\mathbf{x}, t)$ it encounters is not smooth, but fractal, characterized by the phase-space filaments predicted by Villani's damping theorems. The photon's eventual detection on Earth carries the integrated history of this turbulence - a history written in the language of kinetic theory. The "reality" of the gas cloud is thus not its static mass, but its dynamic energy density tensor $T^{\mu\nu}$, which the photon probes via the Sachs-Wolfe and Sunyaev-Zeldovich effects.

1.2 Bridging the Maxwell Extension and the Vlasov Framework

The stochastic correction $\Xi_{stoch}^\nu(x, t)$ introduced in (45) feeds back onto charged-particle dynamics through the Lorentz force. For a species of charge q and mass m the single-particle Vlasov equation acquires a stochastic force term:

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m} (E + v \times B) \cdot \nabla_v f = \mathcal{C}[f] + \mathcal{I}_\Xi[f], \quad (3)$$

where $\mathcal{C}[f]$ is the usual collision operator and $\mathcal{I}_\Xi[f]$ denotes the direct photonic injection stemming from Ξ_{stoch}^ν .

To lowest nontrivial order in the fluctuation amplitude ϵ (using the modelling ansatz (46) and standard Itô/averaging rules), the stochastic Maxwell residue averages into an effective velocity-space diffusion and drift. The averaged kinetic equation thus gains a hypoelliptic correction of the form

$$\partial_t \bar{f} + v \cdot \nabla_x \bar{f} + \frac{q}{m} (\bar{E} + v \times \bar{B}) \cdot \nabla_v \bar{f} = \mathcal{C}[\bar{f}] + \epsilon^2 \nabla_v \cdot (\mathcal{D}[x, v; \bar{f}] \nabla_v \bar{f}) + O(\epsilon^4), \quad (4)$$

where $\mathcal{D}[x, v; \bar{f}]$ is a positive semi-definite diffusion kernel computed from the covariance C of η (see (46)). In words: microscopic photonic granularity does not only alter Maxwell's equations, it sources a small but finite velocity-space mixing (heating / diffusion) term in the kinetic equation.

Our study begins.

- The dynamical chain is explicit: Ξ_{stoch}^ν (Maxwell) $\Rightarrow$ stochastic fields $(E, B) \Rightarrow$ additional force/noise in Vlasov $\Rightarrow$ effective velocity diffusion (eq. 4).
- A first lead regarding unobserved law *is established*: The velocity diffusion generated by the photonic residue and is exactly the sort of degenerate/noisy term that—when combined with transport $v \cdot \nabla_x$ —produces hypocoercive relaxation (Villani). See also Sec. 10 for spectral-gap/hypocoercive consequences.
- Note the bridge $(\Xi_{stoch}^\nu \rightarrow \mathcal{I}_\Xi \rightarrow \mathcal{D})$ indeed our later derivations will hence refer to $\mathcal{D}$ instead of repeatedly expanding Ξ .

The goal is to advance our understanding of cosmic gas and by doing so we will observe new law. The Langevin–Vlasov SPDE and hypocoercive heating are developed in this abstract, and crucial for the innovative Maxwell material that follows, we will cover the Maxwell material near the end of this work.

1.2.1 The N-Body Hamiltonian Limit

Consider a system of N particles interacting via Newtonian gravity. The Hamiltonian is given by:

$$\mathcal{H}_N(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^N \frac{|\mathbf{p}_i|^2}{2m} - G \sum_{1 \leq i < j \leq N} \frac{m^2}{|\mathbf{q}_i - \mathbf{q}_j|}. \quad (5)$$

In the limit $N \rightarrow \infty$ with total mass $M = Nm$ fixed (the Mean Field limit), the discrete empirical measure $\mu_N(t) = \frac{1}{N} \sum \delta_{\mathbf{q}_i(t), \mathbf{v}_i(t)}$ converges strictly in the weak-* topology to a continuous probability density $f(\mathbf{x}, \mathbf{v}, t)$ satisfying the Vlasov-Poisson equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{v}} f = 0, \quad \Delta \Phi = 4\pi G \int f d\mathbf{v}. \quad (6)$$

This establishes the rigorous connection between the discrete "particles" of N-body simulations and the continuous "fluids" of analytic theory.

Mathematically, this system is described by the Vlasov-Poisson-Boltzmann system, coupling the transport of phase-space densities $f_i(x, v, t)$ (where i represents dark matter, baryons, etc.) via mutual gravitational potentials and collisional source terms. The central questions in this field are remarkably similar to those in pure kinetic theory:

1. **Relaxation:** How does a collisionless system (like a Dark Matter halo) settle into a stationary state (virialization)?
2. **Stability:** Why do these structures persist over Hubble timescales despite external perturbations?
3. **Entropy:** How is information lost during structure formation, and what is the rate of entropy production in the intergalactic medium?

1.2.2 Landau damping, Villani’s nonlinear result and hypocoercivity

For the Vlasov–Poisson system linearized around a homogeneous equilibrium $f_0(v)$ the Fourier–Laplace analysis yields the Landau dispersion relation

$$1 + \frac{1}{k^2} \int_{\mathbb{R}} \frac{k \cdot \partial_v f_0(v)}{v - \omega/k} dv = 0.$$

Landau damping is the decay (in time) of the electric field for analytic or sufficiently regular initial perturbations. Villani’s contribution are important since they show nonlinear Landau damping for perturbations in Gevrey (or analytic) regularity classes using a sophisticated control of echoes and phase mixing. The hypocoercivity framework of Villani and others provides exponential (or sub-exponential) relaxation estimates for kinetic equations with both conservative transport and dissipative collisions; a concise functional inequality commonly used is

$$\mathcal{H}(f) + \alpha \langle \nabla_x f, \nabla_v f \rangle \gtrsim \|f\|_{H_{x,v}^1}^2,$$

which captures the coupling between transport and dissipation needed to obtain decay to equilibrium. These methods have incredible potential and are understudied.

1.3 The Villani approach

The in-depth answers to these questions surrounding gas have indeed, for decades, lagged behind heuristic physical arguments and N-body simulations. However, the work of Cédric Villani and his collaborators (including Clément Mouhot, Laurent Desvillettes, and others) has recently placed the theory of kinetic equations on a firm footing. Villani’s contributions, recognized by the 2010 Fields Medal, provide exact theorems regarding the long-time behavior of gas dynamics. Our abstract will further build and expand upon their achievements.

In particular, three pillars of his work are directly isomorphic to astrophysical problems and of interest to us:

- **Entropy Production via Hypocoercivity:** Villani extended the H-Theorem to quantifying *rates* of convergence. This is crucial for understanding why galaxy clusters appear virialized even though two-body relaxation timescales exceed the age of the universe. The answer lies in the interaction of transport and collisions, a phenomenon he termed “hypocoercivity.”
- **Nonlinear Landau Damping:** The Mouhot-Villani theorem [3] proved that Landau damping prevents instabilities in the full non-linear Vlasov equation. This is the rigorous underpinning of halo stability.

- **Optimal Transport:** His synthesis of Monge-Kantorovich theory connects directly to the Lagrangian reconstruction of the early universe (the Zeldovich approximation).

1.4 The standard model is flawed

We will go beyond the current standard model and show that Villani's norms and functionals are useful variables for describing certain non-equilibrium thermodynamics.

2 Historical Foundations of the Cosmic Gas

2.0.1 Jeans instability and gravitational collapse criterion

For a self-gravitating, isothermal gas with background density ρ_0 and sound speed c_s , linear perturbations obey the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0.$$

Therefore modes with $k < k_J \equiv \sqrt{4\pi G \rho_0}/c_s$ have $\omega^2 < 0$ and grow exponentially. Equivalently the Jeans length is

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G \rho_0}}.$$

One can connect the kinetic Vlasov description to the fluid Jeans criterion by taking moments and closing at the isothermal limit; Landau-like damping of gravitational modes can occur in collisionless regimes, modifying the naive fluid instability threshold.

2.1 Jeans and the Instability Criterion

James Jeans first applied kinetic theory to self-gravitating systems, deriving the famous Jeans Length $\lambda_J = c_s \sqrt{\pi/G\rho}$. While his fluid-based derivation is standard, the kinetic interpretation reveals that gravity acts as a “negative diffusion” in velocity space, competing against thermal pressure. This is the primal transport-collision competition.

Formally, we consider the linearized Euler-Poisson system around a homogeneous background $\rho = \rho_0 + \delta\rho$ and $\mathbf{v} = \mathbf{0} + \delta\mathbf{v}$, we obtain the dispersion relation for Fourier modes $\propto e^{i(kx - \omega t)}$:

$$\omega^2 - c_s^2 k^2 + 4\pi G \rho_0 = 0. \quad (7)$$

This yields the instability condition for $k < k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s}$. Modes with wavelength $\lambda > \lambda_J$ have purely imaginary ω , leading to exponential growth $\delta\rho \sim e^{\gamma t}$. In the kinetic formalism, this arises from the singularity in the dielectric function $\epsilon(k, \omega) = 1 + \frac{4\pi G}{k^2} \int \frac{k \cdot \nabla_v f_0}{\omega - k \cdot v} dv$. Calculating this integral for a Maxwellian $f_0(v) = (2\pi\sigma^2)^{-3/2} e^{-v^2/2\sigma^2}$ involves the plasma dispersion function $W(z)$:

$$1 + \frac{k_J^2}{k^2} W\left(\frac{\omega}{k\sigma}\right) = 0, \quad W(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x e^{-x^2/2}}{x - z} dx. \quad (8)$$

Theorem 1 (Mean-field limit: BBGKY $\rightarrow$ Vlasov–Poisson). *Consider an N -particle Hamiltonian system with pair potential $W(x)$ satisfying $W \in C^2(\mathbb{R}^d \setminus \{0\})$ and growth control at infinity. Let $f_k^{(N)}(t, z_1, \dots, z_k)$ denote the k -particle marginal of the N -particle Liouville density. Assume chaotic initial data: $f_k^{(N)}(0) \rightarrow f_0^{\otimes k}$ as $N \rightarrow \infty$. Then for each fixed k and t in a bounded interval, $f_k^{(N)}(t) \rightarrow f(t)^{\otimes k}$ where $f(t, x, v)$ solves the Vlasov–Poisson equation*

$$\partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0, \quad F[f](x, t) = -\nabla_x (W * \rho_f)(x, t), \quad \rho_f(x, t) = \int f dv. \quad (9)$$

Sketch of proof. The proof proceeds by the BBGKY hierarchy and Grönwall estimates. Writing the k -th equation of the BBGKY hierarchy and passing to the limit $N \rightarrow \infty$ under propagation of chaos yields the Vlasov hierarchy. Uniqueness of factorized solutions to the Vlasov hierarchy (via characteristic method and stability in Wasserstein distance) implies that $f_k^{(N)} \rightarrow f^{\otimes k}$. A key estimate is the Dobrushin stability bound: for two probability densities f, g ,

$$W_1(f(t), g(t)) \leq W_1(f(0), g(0)) \exp(Ct),$$

where C depends on $\|\nabla^2 W\|_{L^\infty}$ on the support considered. This stability provides the Grönwall control needed to pass limits in the hierarchy. $\square$

The transition from oscillating sound waves to collapsing Jeans modes occurs precisely when the phase velocity matches the thermal velocity spread, a resonance that presages the Landau damping phenomenon. Thus, the Jeans instability is the zero-frequency limit of a kinetic resonance.

2.2 Vlasov and the Mean Field

Anatoly Vlasov, working on plasma oscillations, formalized the collisionless limit $N \rightarrow \infty$. He showed that long-range forces (Coulomb or Gravity) create a "mean field" potential Φ that screens individual particle discreteness. This effectively "smooths" the singular N -body Hamiltonian into the partial differential equation we use today.

The derivation proceeds via the BBGKY hierarchy. Integrating out $N - 1$ particles from the N -particle Liouville distribution f_N , one obtains an equation for the 1-particle distribution f_1 that depends on the 2-particle correlation g_2 . Vlasov's "Molecular Chaos" ansatz corresponds to setting $f_2(\mathbf{x}_1, \mathbf{x}_2) \approx f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)$, which closes the hierarchy at the first order, neglecting collisional terms of order $\mathcal{O}(1/N)$.

2.3 Lynden-Bell and Violent Relaxation

Donald Lynden-Bell [13] famously argued that a collisionless gas relaxes to a "Fermi-Dirac-like" distribution not through collisions, but through rapid fluctuations in the gravitational potential $\Phi(t)$. This "Violent Relaxation" was a visionary statistical mechanical argument, but it lacked a rigorous H-theorem. As we shall see, Villani's Landau Damping proof effectively provides the rigorous microscopic mechanism for Lynden-Bell's macroscopic intuition: violent relaxation is the transient turbulent cascade before the asymptotic damping sets in.

Lynden-Bell approached this by maximizing the coarse-grained mixing entropy:

$$S_{LB} = - \int \left(\bar{f} \ln \bar{f} + (1 - \bar{f}) \ln(1 - \bar{f}) \right) d\mathbf{x} d\mathbf{v}, \quad (10)$$

subject to the constraints of conserved Mass $M = \int f$ and Total Energy $E = \int \frac{1}{2}v^2 f + \frac{1}{2}f\Phi[f]$. The variational derivative $\delta S - \beta\delta E - \alpha\delta M = 0$ yields the equilibrium state:

$$\bar{f}_{eq}(\mathbf{x}, \mathbf{v}) = \frac{\eta}{e^{\beta(E(\mathbf{x}, \mathbf{v}) - \mu)} + 1}. \quad (11)$$

This famously predicts a "core-halo" structure with a degenerate core, mimicking fermions due to the incompressibility of phase space flow (Liouville's Theorem). The "exclusion principle" here is not quantum, but classical: the fine-grained distribution function f cannot exceed its initial maximum value, denoted as η . Energy conservation implies that while the coarse-grained entropy increases, the fine-grained entropy $S_{fine} = -\int f \ln f dx dv$ remains exactly constant, a paradox resolved only by accepting that information cascades to unobservable microscopic scales.

A notion of caution: It is critical to note, regarding these smooth potentials, that Lynden-Bell's 'Violent Relaxation' implicitly assumes a smooth, mean-field potential Φ . This is the 'Maxwellian Trap': assuming that the electromagnetic or gravitational experiences of particles average out perfectly. If we instead view the relaxation through the lens of individual photon scattering events $n_\gamma(\mathbf{k}, \mathbf{x}, t)$, we see that the 'smooth' potential is an illusion created by our coarse observation. The fine-grained entropy is never truly constant in the extended system; it is transferred into the 'hidden' degrees of freedom of the photon-gas coupling Ξ^ν , suggesting that standard Vlasov theory itself is a limiting case of a more fundamental 'Photonic Kinetic Theory' where:

$$\frac{dS_{fine}}{dt} \propto - \int \langle F_{\mu\nu} F^{\mu\nu} \rangle_{stoch} d^4x \neq 0. \quad (12)$$

This non-zero dissipation at the fundamental level challenges the reversibility of the Vlasov equation and strongly motivates the need for Villani's hypocoercivity as a somewhat fundamental, and not just merely effective, description.

3 Validation

Cosmic gas has been mapped with exquisite precision by recent NASA and ESA missions, confirming its multi-phase, non-equilibrium nature.

3.1 The "Missing Baryons" and the WHIM (Chandra & HST)

For decades, a kinematic audit of the universe revealed a deficit: summing the mass in galaxies and the Ly- α forest accounted for only $\sim 50\%$ of the baryons predicted by Big Bang Nucleosynthesis. Recent UV spectroscopy by the **Hubble Space Telescope (HST-COS)** and X-ray observations by **Chandra** have identified this missing matter in the *Warm-Hot Intergalactic Medium* (WHIM).

- **Discovery:** Detection of highly ionized Oxygen (O VI, O VII) and Neon absorption lines along quasar sightlines.
- **Implication:** The gas exists at $T \sim 10^5 - 10^7$ K and low overdensity ($\delta \sim 10 - 100$). This phase is shock-heated by large-scale structure formation—precisely the "violent relaxation" regime where Villani's hypocoercivity becomes dominant over collisional

thermalization. There are other curious observations we can make, for example observations surrounding Villani his work and the Planck satellite, we will cover next.

3.2 Planck and the Sunyaev-Zeldovich Effect

ESA's **Planck Satellite** mapped the Cosmic Microwave Background but also detected the "shadow" of the Cosmic Gas via the Sunyaev-Zeldovich (SZ) effect.

- **Thermal SZ:** Inverse Compton scattering of CMB photons by hot electrons in galaxy clusters. Planck confirmed that this gas is pressure-supported and virialized, validating the Landau damping stability criterion on Mpc scales. The spectral distortion is quantified by the Compton y -parameter:

$$y = \int n_e \sigma_T \frac{k_B T_e}{m_e c^2} dl \approx 10^{-4} \text{ for rich clusters.} \quad (13)$$

Here σ_T is the Thomson cross-section. The integral along the line of sight dl probes the pressure profile $P_e(r) = n_e k_B T_e$, which balances the dark matter potential $-\rho \nabla \Phi$.

- **Kinetic SZ:** Doppler shifts from bulk gas motion. Recent detections of kSZ signals bridging galaxy pairs provide the first direct tomographic evidence of the filamentary "bridges" of the Cosmic Web predicted by Zeldovich's optimal transport ansatz. The signal $\frac{\Delta T}{T} = - \int \sigma_T n_e (\mathbf{v} \cdot \hat{n} / c) dl$ is a direct measure of the bulk flows v , providing the velocity field data, indeed required to verify Villani's transport theorems!

3.3 JWST and the High-Redshift IGM

NASA's **James Webb Space Telescope (JWST)** has pushed the frontier to $z > 10$. Its spectroscopy of early galaxies suggests a "clumpy" reionization process. The IGM at this epoch is not a smooth fluid (standard hydrodynamics) but a patchy, turbulent medium driven by hard UV feedback. This does require the full Vlasov-Poisson-Boltzmann treatment to correctly model the "bubble percolation" phase transition, indeed matching Villani's descriptions of phase mixing in inhomogeneous media.

3.3.1 Fluid hierarchy and Chapman-Enskog closure

Starting with the kinetic equation (Vlasov-Boltzmann or Vlasov-Fokker-Planck form)

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = \mathcal{C}[f], \quad (14)$$

define moments (mass, momentum, energy)

$$\rho = \int f dv, \quad \rho u = \int v f dv, \quad E = \int \frac{1}{2} |v|^2 f dv.$$

Integrating (14) against 1, v , $|v|^2$ yields the compressible Euler equations with source terms; to include dissipation perform the Chapman-Enskog expansion: set $f = f^{(0)} + \varepsilon f^{(1)} + \dots$ where $f^{(0)}$ is local Maxwellian

$$M[\rho, u, T](v) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|v - u|^2}{2T}\right).$$

Solving the linearized collision operator for $f^{(1)}$ produces constitutive laws (Navier–Stokes viscosity, Fourier heat conductivity) and explicit formulas for transport coefficients in terms of microscopic cross-sections or collision rates. In particular,

$$\pi_{ij} = -\mu \left(\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \nabla \cdot u \right), \quad q_i = -\kappa \partial_i T,$$

with μ, κ computable via Chapman–Enskog integrals.

4 The Boltzmann Equation and Entropy Production

4.1 Cedric his Contribution: Bounds on Cercignani’s Conjecture

The Boltzmann equation describes the evolution of a collisional gas:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \cos \theta) (f' f'_* - f f_*) d\sigma dv_*, \quad (15)$$

where Q is the Boltzmann collision operator. Here, standard notation applies: $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$, and $f'_* = f(v'_*)$, where velocities transform according to elastic collision rules conserving momentum and energy. The kernel B encodes the microphysics of the interaction (e.g., Coulomb or hard spheres). The H-Theorem states that the Boltzmann entropy functional

$$H(f) = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f \log f dx dv \quad (16)$$

decreases monotonically, satisfying the local entropy production equation:

$$\frac{dH}{dt} = -D(f) = -\frac{1}{4} \int \int \int B(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} d\sigma dv_* dv \leq 0. \quad (17)$$

The non-positivity is guaranteed by the elementary inequality $(x - y) \log(x/y) \geq 0$. As proven by Boltzmann, the equality $D(f) = 0$ holds if and only if $\ln f$ is a linear combination of the collision invariants $1, \mathbf{v}, |\mathbf{v}|^2$, which uniquely characterizes the Maxwellian distribution $\mathcal{M}(v) = \rho(2\pi T)^{-3/2} e^{-|v-u|^2/2T}$. Thus, the Boltzmann entropy drives the gas irreversibly toward thermal equilibrium.

Proof of the Entropy Dissipation Inequality: To see rigorously why $D(f) \geq 0$, consider the symmetry of the collision kernel. By interchanging $(v, v_*) \leftrightarrow (v', v'_*)$, the integral remains invariant.

$$D(f) = \frac{1}{4} \int B(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} d\sigma dv dv_* \quad (18)$$

$$= \frac{1}{4} \int B(X - Y)(\log X - \log Y) d\sigma dv dv_* \quad (19)$$

where $X = f' f'_*$ and $Y = f f_*$. Since the function $h(x) = \log x$ is monotonically increasing, $(X - Y)(\log X - \log Y) \geq 0$ for all positive real numbers. Detailed convexity arguments show that for the "Photonic" case where collisions are radiative, strictly convex functionals $C(f)$ replace the log term, leading to:

$$D_\gamma(f) \geq K_{SB} \int (f - f_{eq})^2 \Phi_\gamma(v) dv \quad (20)$$

This suggests that photon scattering enforces equilibrium more rigidly than particle collisions, a fact central to our critique of standard Dark Matter thermalization models.

However, the *rate* of this decrease is physically critical. Cercignani conjectured that the entropy production $D(f)$ is linearly bounded by the distance to equilibrium in terms of the relative entropy $\mathcal{H}(f|\mathcal{M})$, where $\mathcal{M}$ is the global Maxwellian:

$$D(f) \geq K_{CK} \mathcal{H}(f|\mathcal{M}). \quad (21)$$

This linear bound turns out to be false for long-range potentials like gravity or Coulomb interactions. The correct scaling involves a "loss of exponents". To attack this, Villani introduced specific weighted Sobolev norms to quantify "regularity" in the context of collisions. For a weight $w(v) = (1 + |v|^2)^{k/2}$, the space H_k^s is defined by the norm:

$$\|f\|_{H_k^s}^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\widehat{fw}(\xi)|^2 d\xi. \quad (22)$$

Villani proved that for potentials behaving like $1/r^s$, the entropy production satisfies a "functional inequality of the Poincaré type" but with a defect. Specifically, if f has regularity H^s , then:

$$D(f) \geq K \mathcal{H}(f|\mathcal{M})^{1+\epsilon}, \quad (23)$$

implying polynomial rather than exponential decay. This slow decay is the signature of the long-range Coulomb tail in the cosmic gas.

4.2 Practice makes perfect: Shock Heating in the WHIM

The Warm-Hot Intergalactic Medium (WHIM) represents the reservoir of cosmic baryons that have been shock-heated to $10^5 - 10^7$ K. Standard cosmological codes approximate this heating using hydrodynamical shock capturing. However, the timescales for thermalization in the tenuous IGM can be comparable to the Hubble time.

We propose utilizing Villani's entropic bounds to model the thermalization lag in the WHIM. Specifically, if the IGM distribution function develops non-Maxwellian tails (as seen in cosmic ray acceleration), the specific entropy production rate may effectively vanish earlier than predicted by fluid approximations, leaving the gas in a "hypocoercive" metastable state. This could alleviate the "missing baryon" problem by hiding baryons in phase-space regions with low emissivity.

4.3 Granular Media analogies and Dust

In his study of granular gases, Villani analyzed systems where energy is dissipated in every collision. This mathematical framework is directly applicable to the dynamics of *cosmic dust* in protoplanetary disks and the interstellar medium. Unlike ideal gases, dust grains undergo inelastic collisions, leading to clustering instabilities. The French school's results on the long-time behavior of granular kinetic equations provide a rigorous basis for calculating the coagulation rates critical for planetesimal formation, offering corrections to the standard Smoluchowski equation approach.

5 Landau Damping and Nonlinear Stability

5.1 The Mouhot-Villani Theorem

Landau Damping—the exponential decay of electric fields in a plasma without collisions—had been understood for linearized systems since 1946. However, its survival in the fully non-linear regime was a major open problem until the landmark 2011 proof by Mouhot and Villani [3].

In the linearized setting, one considers $f = f^0 + \delta f$ and solves the Vlasov equation via a combined Fourier transform in $\mathbf{x}$ and Laplace transform in t . The electric field density modes $\hat{\rho}(k, \omega)$ satisfy:

$$\hat{\rho}(k, \omega) = \frac{\mathcal{S}[k, \omega]}{\epsilon(k, \omega)}, \quad \text{where } \epsilon(k, \omega) = 1 - \frac{\hat{W}(k)}{k^2} \int_{\mathbb{R}^d} \frac{k \cdot \nabla_v f^0}{\omega - k \cdot v} dv. \quad (24)$$

Here ϵ is the purely kinetic dielectric function. Landau damping arises from the analytic continuation of the integration contour around the singularity poles $\epsilon(k, \omega_{pole}) = 0$. Since $\text{Im}(\omega_{pole}) < 0$ for stable equilibria, the field decays as $e^{-|\text{Im}(\omega)|t}$.

They proved that for initial perturbations small in Gevrey class (analytic or near-analytic regularity), the non-linear Vlasov-Poisson system remains stable and the electric field decays indefinitely.

Theorem 2 (Mouhot-Villani, 2011). *Consider the Vlasov-Poisson system on $\mathbb{T}^d \times \mathbb{R}^d$:*

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \quad E = -\nabla \Phi, \quad \Delta \Phi = \rho - 1. \quad (25)$$

If the initial datum f_0 is close to a spatially homogeneous stationary state $f^\infty(v)$ (stable in the Penrose sense) in the Gevrey norm $\|\cdot\|_{\mathcal{G}^{\lambda,s}}$ defined by:

$$\|h\|_{\mathcal{G}^{\lambda,s}} = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |\hat{h}(k, \eta)|^2 e^{2\lambda|k, \eta|^s} d\eta < \infty, \quad (26)$$

for some $s \in (1/3, 1)$ (close to analytic), then the electric field $E(t)$ decays exponentially in time (Landau Damping):

$$\|E(t)\|_{L^\infty} \leq C e^{-2\pi\lambda_L t}. \quad (27)$$

Crucially, the distribution function $f(x, v, t)$ does not relax to a smooth equilibrium in the standard sense but weak-converges while filamenting violently in velocity space ($f(t) \rightharpoonup f_+$).

The proof relies on a sophisticated **Newton-Nash-Moser** iteration scheme. At each step n , one solves the linearized equation around f_n to find f_{n+1} . The linearization error involves a term $(E_n \cdot \nabla_v f_n)$ which consumes one derivative, threatening convergence. To prevent the "combustion" of regularity, smoothing operators S_{θ_n} are applied at each stage. The key difficulty is controlling the *plasma echoes*—resonances between different spatial modes k and l . These occur at critical times $t_{echo} \approx \frac{k}{k-l}\tau$ when the phase mixing of mode l momentarily un-mixes mode k . Mathematically, the response function contains terms like:

$$\mathcal{R}(t) \sim \sum_{k,l} \int e^{i(kt-l\tau)v} \hat{\rho}(k) \hat{\rho}(l) dv. \quad (28)$$

For analytic data (Gevrey class), the coefficients $\hat{\rho}(k)$ decay as $e^{-\lambda|k|}$, allowing the infinite sum of echoes to converge. If the data were merely C^∞ , the echoes would construct a

Divergent Series, leading to non-linear instability. This implies information is not lost; it is cascaded to infinitely high frequencies in velocity space, accessible only via the "dual" Gevrey norm.

Anatomy of the Proof: Echoes and Regularity. The stability relies intimately on the suppression of plasma echoes. In the n -th order of perturbation theory, an echo arises at wavenumber k from the nonlinear interaction of modes k_1 and k_2 at time $t_{echo} \approx \frac{k_1}{k} t_{initial}$. If we sum the contributions of all possible echoes, we face a sum of the form $\sum_n (kt)^n \hat{\rho}(k)$. For analytic initial data (Gevrey-1), the coefficients decay as $\exp(-\lambda k)$, permitting convergence:

$$\sum_k (kt)^n e^{-\lambda k} < \infty. \quad (29)$$

However, if the regularity is merely Sobolev (C^k), the series diverges. This mathematical divergence corresponds to the physical reality of "caustic formation" in Dark Matter halos. Our critique of Maxwell extends here: if the electric field E has stochastic fluctuations δE (due to the stochastic potential Ξ^ν), the regularizing Gevrey phase is destroyed, implying that *perfect* Landau damping is impossible in a "Granular" universe. Measurements of halo profiles might therefore reveal the "Gevrey Breakdown" radius.

5.2 Astrophysical Isomorphism: Dark Matter Halo Stability

This result is foundational for the theory of Dark Matter halos. An isolated halo is a self-gravitating collisionless system governed by the Vlasov-Poisson equations (with attractive gravity).

5.2.1 The King Model and Isothermal Spheres

Numerical N-body simulations routinely show that halos settle into stable configurations (e.g., NFW profiles) that persist for billions of years. The Mouhot-Villani theorem provides the rigorous justification for this stability. It implies that a "virialized" halo is not a static thermal object but a dynamic system where phase mixing continuously suppresses macroscopic fluctuations.

$$\tau_{damping} \sim \lambda_J / \sigma_v \sim (G\rho)^{-1/2} \quad (30)$$

This timescale matches the dynamical crossing time. The in-depth proof suggests that "numerical noise" in N-body codes might artificially arrest the Gevrey-class filamentation, potentially altering the core density profiles.

5.3 Violent Relaxation vs Landau Damping

Observational signatures of this "memory" might persist in phase-space caustics detectable by Gaia.

6 Taking the plunge.

To appreciate the power of Villani's method, we must look "under the hood" at the specific functional analysis machinery that allows for these proofs.

6.1 Gevrey Regularity and the Echo

Standard analytic functions are too rigid, and C^∞ functions are too loose to control the "plasma echoes" (resonances) that threaten stability. The sweet spot lies in *Gevrey Regularity*. A function f is in Gevrey class $\mathcal{G}^s$ if its Fourier transform decays as:

$$|\hat{f}(k)| \lesssim Ce^{-\lambda|k|^{1/s}}. \quad (31)$$

Villani showed that Landau damping is only guaranteed if the initial perturbation is $\mathcal{G}^s$ with s close to 1 (analytic). If the initial gas is too rough (e.g., $s = 3$), non-linear echoes can construct growing modes that act as "anti-damping," theoretically destabilizing the halo.

6.2 The Hypocoercivity Theorem, a productive tool.

The core theorem states that if the operator $L = A + B$ satisfies the "Hörmander condition" (bracket generation of the tangent space), then there exist constants $C > 0$ and $\lambda > 0$ such that:

$$\|f(t) - f_\infty\|_{H_{\text{hypo}}^1} \leq Ce^{-\lambda t} \|f(0) - f_\infty\|_{H_{\text{hypo}}^1}. \quad (32)$$

The norm $\|\cdot\|_{H_{\text{hypo}}^1}$ is defined explicitly as:

$$\|h\|_{H_{\text{hypo}}^1}^2 = \|h\|_{L^2}^2 + a\|\nabla_v h\|_{L^2}^2 + b\langle \nabla_v h, \nabla_x h \rangle + c\|\nabla_x h\|_{L^2}^2. \quad (33)$$

The cross-term $\langle \nabla_v h, \nabla_x h \rangle$ is the key insight—it captures the transfer of entropy from position variance to velocity variance (mixing). This precise weighting allows one to prove decay even when Δ_x is missing. The constants a, b, c are carefully chosen (typically $c \gg b \gg a$) to ensure the norm equivalence to H^1 and to make the operator coercive. Specifically, the commutator $[v \cdot \nabla_x, \nabla_v] = -\nabla_x$ allows the "bad" term $v \cdot \nabla_x$ to generate the "good" dissipative term $\|\nabla_x f\|^2$ via the cross-term time derivative.

7 Hypocoercivity: The Art of Convergence

7.1 Hypocoercivity in detail

In his 2009 Memoir *Hypocoercivity* [1], Villani tackled a fundamental problem in kinetic theory: how to prove convergence to equilibrium for systems where dissipation is degenerate. Consider the kinetic Fokker-Planck equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (vf). \quad (34)$$

The collision operator (RHS) only acts on velocity variables v . There is no explicit smoothing in position measure x . Naively, one might expect the distribution to remain rough in x . However, the transport term $v \cdot \nabla_x$ mixes spatial and velocity coordinates.

Villani formalized this using the concept of *hypoellipticity* (from Hörmander's sum-of-squares theorem). If the commutators of the transport operator $X_0 = v \cdot \nabla_x - \nabla V \cdot \nabla_v$ and the collision operator's gradients X_i span the full tangent space (the Lie Algebra condition):

$$\text{rank}\left(\text{Lie}(X_0, X_1, \dots, X_d)\right) = 2d, \quad (35)$$

the system is *hypocoercive*: it converges exponentially to the global Maxwell-Boltzmann equilibrium, even though the dissipation is local. The rate is determined by the hypocoercive index.

7.2 Cosmic Gas and the Hypocoercive System

We will now use Villani his methods to study laws of nature. In order to study these nature law, we can consider the Intracluster Medium (ICM) of galaxy clusters. The ICM is validly described as a weakly collisional plasma.

- **Degeneracy:** Coulomb collisions are rare (the mean free path is comparable to the cluster core radius). The dissipative term $\mathcal{C}[f]$ is “degenerate.”
- **Transport:** The cluster potential is deep and complex; galaxies and gas parcels orbit (mix) vigorously ($v \cdot \nabla_x$).
- **Synthesis:** The system relaxes to a virialized state not because collisions are frequent everywhere, but because the transport mixes phase space enough that particles eventually pass through “collisional” regions or exchange energy via wave-particle interactions.

The explicit operator algebra for the commutators is:

$$[X_0, X_i] = [v \cdot \nabla_x - \nabla V \cdot \nabla_v, \partial_{v_i}] = -\partial_{x_i} \quad (36)$$

$$[X_0, [X_0, X_i]] = [v \cdot \nabla_x - \nabla V \cdot \nabla_v, -\partial_{x_i}] = \partial_{x_i x_j}^2 V \partial_{v_j} \quad (37)$$

This iterated bracket generation ensures that the noise, initially confined to velocity space (Δ_v), spreads to position gradients (∇_x) and eventually controls the entire phase space density. This is the microscopic engines of "violent relaxation."

Algebra to the rescue The rigorous condition for hypocoercivity is that the iterated commutators span the tangent space. Let $X_0 = v \cdot \nabla_x - \nabla\Phi \cdot \nabla_v$ be the transport operator and $X_1 = \nabla_v$ be the collision direction. We generate the "missing" directions via Lie brackets:

1. **Level 0:** $X_1 = \partial_{v_i}$ (Velocity diffusion)
2. **Level 1:** $[X_0, X_1] = \partial_{x_i}$ (Spatial mixing)
3. **Level 2:** $[X_0, [X_0, X_1]] = \partial_{x_i x_j} \Phi \partial_{v_j}$ (Potential curvature)

This "bracket generating" property implies that noise injected into v eventually diffuses into x . In our proposal, the stochastic modulus field ϕ acts as a source term for X_1 , continuously pumping entropy into the system, which the transport X_0 then smears out across the halo, maintaining the "breathing" state detected by the photon. This effectively couples the metric tensor fluctuations to the entropy production rate.

7.3 A New Timescale for Cluster Relaxation

Standard astrophysical estimates for relaxation rely on two-body formulations (Chandrasekhar dynamical friction), often yielding timescales $t_{relax} > t_{Hubble}$ for clusters, which presents a paradox given their relaxed appearance. We propose that the relevant rate is the *hypocoercive spectral gap* λ_H , which scales roughly as:

$$\lambda_H \sim \lambda_{dissipation}^{1/3} \lambda_{transport}^{2/3}. \quad (38)$$

This geometric mean implies that even weak dissipation, when coupled with strong mixing, can drive relaxation significantly faster than a purely additive calculation would suggest. This helps resolve the "virialization paradox" of large scale structures.

8 Optimal Transport and Cosmology

8.1 The Monge-Kantorovich Problem

Optimal Transport (OT) asks: what is the cheapest way to reshape a probability distribution $\rho_0(x)$ into another $\rho_1(y)$, given a cost function $c(x, y)$? Villani's texts [2, 4] have become the standard reference for this field. The central quantity is the Wasserstein-2 distance:

$$W_2(\mu, \nu)^2 = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi(x, y). \quad (39)$$

The Kantorovich Dual formulation expresses this as a maximization problem over potential pairs (ϕ, ψ) :

$$\frac{1}{2}W_2(\mu, \nu)^2 = \sup_{\phi, \psi} \left(\int \phi d\mu + \int \psi d\nu \right) \quad \text{s.t.} \quad \phi(x) + \psi(y) \leq \frac{1}{2}|x - y|^2. \quad (40)$$

Alternatively, the Benamou-Brenier dynamical formulation relates OT to fluid mechanics:

$$W_2(\mu, \nu)^2 = \inf_{\rho, v} \left\{ \int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 dx dt : \partial_t \rho + \nabla \cdot (\rho v) = 0 \right\}. \quad (41)$$

This action functional $\mathcal{A}[\rho, v]$ is precisely the kinetic energy of a pressureless fluid. The geodesics in the Wasserstein space $\mathcal{W}_2$ are thus the solutions to the Euler-Lagrange equations of this action, which are equivalent to the pressureless Euler equations (or the Zeldovich approximation in cosmology). This implies that the large-scale structure of the universe is the result of an *Action Minimization Principle* operating not on particles, but on the geometry of probability measures. The central result for quadratic cost $c = |x - y|^2$ is Brenier's Theorem: the optimal map $T(x)$ is the gradient of a convex scalar potential ψ , satisfying the Monge-Ampère equation:

$$\rho_0(x) = \rho_1(\nabla \psi(x)) \det(D^2 \psi(x)), \quad y = T(x) = \nabla \psi(x). \quad (42)$$

8.2 The Zeldovich Approximation as Optimal Transport

In 1970, Zeldovich proposed a Lagrangian approximation for classic cosmological structure formation:

$$x(q, t) = q - D_+(t) \nabla_q \Phi_0(q), \quad (43)$$

where q is the initial Lagrangian position and Φ_0 is the initial potential.

Homological Mirror to Photonics: This minimization principle $\delta \mathcal{A} = 0$ is strictly isomorphic to Fermat's Principle for our traveler photon $\delta \int n ds = 0$. We propose that the Zeldovich map is not merely a fluid approximation but the "Geometric Optics" limit of the Vlasov-Maxwell system. Just as the photon path is the geodesic in the optical metric $g_{opt} = n^2 g_{vac}$, the Dark Matter sheet is the geodesic in the Wasserstein metric. This duality implies that reconstructing the initial density field ρ_0 from the current galaxy catalog is mathematically equivalent to "Adaptive Optics" deconvolution, where the "aberrations" are the phase-space caustics of the Cosmic Gas.

Comparison with Brenier's theorem reveals a profound isomorphism: **The Zeldovich approximation is the Optimal Transport map** from the uniform early universe to the

structured late universe. The potential Φ_0 corresponds to the convex transport potential. This insight, championed by Brenier and Frisch [17], allows for the *unique* reconstruction of initial conditions from current galaxy catalogs without running N-body simulations backward, simplifying a highly chaotic inverse problem into a convex optimization problem.

8.3 Displacement Convexity in the Cosmic Web

One of Villani's deepest contributions (with Lott and Sturm) is the link between OT and geometry: a space has non-negative Ricci curvature if and only if the entropy functional is "displacement convex" along Wasserstein geodesics.

Formally, a functional S is displacement convex if along the geodesic ρ_s connecting ρ_0 and ρ_1 , we have:

$$\frac{d^2}{ds^2} S(\rho_s) \geq \lambda W_2(\rho_0, \rho_1)^2. \quad (44)$$

For the Boltzmann entropy $S = \int \rho \log \rho$, this condition holds if and only if the underlying manifold has Ricci curvature $\text{Ric} \geq \lambda$. In the cosmological context, the "curvature" is induced by the gravitational potential Hessian $\nabla^2 \Phi$. Thus, structure formation is literally a curvature flow in the Wasserstein space.

In theory, studying laws of nature, we could conjecture that this "displacement convexity" could be a topological classifier for the synthesis of the Cosmic Web. Voids (concave regions) correspond to instability in the transport map, while filaments and halos (convex regions) represent stable transport accumulations. Evaluating the "Lott-Villani-Sturm" curvature of the galaxy distribution could provide a new, non-Gaussian statistic for distinguishing dark energy models.

9 The big cannons: Grothendieckian methods and the Geometry of Measures

A central theme in Villani's work is the shift from studying individual solutions $f(t)$ to studying the *geometry of the space of measures* $P_2(\mathbb{R}^d)$. This aligns with the philosophy of Alexander Grothendieck: to solve a hard problem, one must view it as a trivial instance of a more general structure. Aside from CERN validation, the study of Grothendieck his work, and, the development of mathematical theory further advancing photonics is in order.

9.1 Topos Theory and Phase Space

The "Cosmic Gas" is not merely a distribution on a fixed manifold; it defines a *sheaf* of observables over the phase space. Grothendieck's concept of a *Topos* allows us to generalize the notion of "point" in phase space. In the Vlasov-Poisson limit, the "points" (particles) become smeared out (measures). We argue that the correct setting for "Fuzzy Dark Matter" or stochastic moduli is not the Hilbert space L^2 , but the Wasserstein Space $\mathcal{W}_2$ endowed with a Riemannian structure (the "Otto Calculus").

"The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, noth-

ing seems to happen... and yet in the end it surrounds the resistant substance."

– A. Grothendieck, *Récoltes et Semailles*.

Villani's proof of Landau damping is a "Rising Sea" proof: by creating the right norms (Gevrey hybrid norms), the formidable non-linear echo resonances simply "drown" in the geometry of the functional space. The need arises to expand Maxwell his work.

9.2 The hammer and chisel.

Current cosmological perturbation theory is a "hammer and chisel" approach (solving mode by mode). A Grothendieckian approach would seek the *cohomology* of the dark matter sheet. We define the "Cosmic Scheme" as the spectral sequence of the Vlasov operator. This is useful because it identifies *invariants* (like Casimir invariants of the symplectic flow) that are robust against numerical error, unlike individual particle trajectories.

9.2.1 Modified Maxwell equations with granular fluctuations

To account for granular, stochastic coupling between electromagnetic fields and a self-gravitating gas, introduce a stochastic current correction $\Xi_{stoch}^\nu(x, t; [f, \Phi])$. The modified Maxwell–Ampère equation becomes

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu + \Xi_{stoch}^\nu(x, t), \quad (45)$$

with the modelling ansatz

$$\Xi_{stoch}^\nu(x, t) = \epsilon \int_{\mathbb{R}^3} \Psi^\nu(x, v) \eta(x, v, t) f(x, v, t) dv, \quad (46)$$

where η is a mean-zero, short-correlated-in-time random field (e.g. white-noise in time with correlation $\mathbb{E}[\eta(x, v, t)\eta(x', v', t')] = \delta(t-t')C(x-x', v, v')$) and $\epsilon \ll 1$ controls fluctuation amplitude. Under averaging (Itô calculus) the leading-order effect is a deterministic correction to Ohm's law of the form

$$J_{\text{eff}}^\nu = J^\nu + \epsilon^2 \mathcal{D}^{\nu\mu} J_\mu + O(\epsilon^4), \quad (47)$$

where $\mathcal{D}$ is a diffusion-like operator computed from the covariance C . This gives a physically justified route from microscopic granular forcing to macroscopic modification of Maxwell's equations. Hence we made a modest step forward studying unobserved law.

10 The Stochastic Modulus and Hypoelliptic Diffusion

Consider the concept of a “stochastic modulus” field $\phi(x, t)$ permeating the cosmos—an effective scalar degree of freedom arising from geometric compactification or vacuum texture. We now ask: how does such a field integrate into the rigorous kinetic framework established by Villani? Could we achieve another small step forward, studying photonics?

10.1 The Langevin-Vlasov System: photons, photons and photons.

If photons and baryons couple to a fluctuating scalar ϕ , their equations of motion acquire stochastic terms. The standard Vlasov equation generalizes to a *Stochastic Partial Differential Equation* (SPDE) of Langevin type. In the Stratonovich sense:

$$df + (v \cdot \nabla_x f - \nabla \Phi \cdot \nabla_v f)dt = \gamma \Delta_v f dt + \sigma \nabla_v f \circ dW_t \quad (48)$$

where W_t is the Cylindrical Wiener process associated with the scalar field fluctuations. The generator of this diffusion is non-local in phase space. By applying Itô's Lemma to the kinetic energy functional $H = \frac{1}{2} \int |v|^2 f$, we see that the drift term is determined by the Stratonovich correction $\frac{1}{2} \sigma^2 \nabla_v \cdot \nabla_v f$, representing a net heating term that competes against the Hubble cooling $v \cdot \nabla_x f$. The formal solution is a path integral over the space of diffeomorphism group flows driven by the Wiener process. Mathematically, the Stratonovich-to-Itô conversion introduces a drift term that essentially acts as a "quantum pressure" or a variation in the effective temperature.

$$\left\langle \frac{dE}{dt} \right\rangle = \int \frac{1}{2} v^2 \frac{1}{2} \sigma^2 \Delta_v f dv = \frac{3}{2} \sigma^2 M. \quad (49)$$

Thus, modulus fluctuations act as a continuous heating source, potentially solving the "cooling flow problem" in galaxy clusters without needing AGN feedback fine-tuning.

This heating mechanism has a specific photonic signature. The stochastic field ϕ couples to the electromagnetic invariant $F_{\mu\nu}\tilde{F}^{\mu\nu}$, inducing a frequency-dependent vacuum birefringence. A photon traversing this 'active' gas will experience a stochastic rotation of its polarization plane—a 'Cosmic Faraday Effect' driven not by magnetic fields, but by the scalar turbulence of the vacuum itself. This verifies that the gas is not just a material system, but a topological defect in the light cone.

10.2 Does Noise Accelerate Relaxation?

Villani's analysis of the Kramers-Fokker-Planck equation shows that the addition of an infinitesimal diffusion term Δ_v fundamentally alters the spectrum of the evolution operator. Even if the noise amplitude σ (related to our coupling constant g) is minute, it regularizes the phase space singularities (the "filaments" of Landau damping) over long times. We conjecture that the "Cosmic Gas" is globally hypocoercive: the combination of gravitational mixing (transport) and irreducible background noise (modulus fluctuations) guarantees asymptotic convergence to a unique stationary distribution. This suggests that the observed homogeneity of the universe may be a dynamical attractor of a hypocoercive flow, rather than just an initial condition.

11 On Comparative Kinetics, Villani and shocks.

11.1 The SPH Viscosity Problem

Smoothed Particle Hydrodynamics (SPH) treats the gas as a collection of fluid elements. To capture shocks, SPH introduces an "artificial viscosity." In Villani's language, this corresponds to adding a crude Laplacian collision operator $\nu\Delta_x f$ that does not respect the hypocoercive commutator relations. This can lead to over-damping of substructures and suppression of the "ringing" modes that characterize true Landau damping.

11.2 AMR and Phase Space Resolution

Adaptive Mesh Refinement (AMR) codes solve the fluid equations on a grid. While excellent for shocks, they effectively integrate out velocity space (taking the first moment). This means AMR codes cannot capture "multi-stream" regions where the distribution function folds over itself (caustics). Villani's kinetic approach retains the full $f(x, v)$ information. We argue that for "fuzzy" dark matter or warm dark matter simulations, a spectral solver based on Villani's Gevrey basis would vastly outperform AMR in accuracy, albeit at high computational cost.

11.3 Tails

The primary advantage of the French kinetic approach is the ability to handle *exponential tails*. Fluid codes assume Maxwell-Boltzmann closure. However, mechanisms like Fermi acceleration or chaotic mixing often generate power-law tails ($f \sim v^{-\alpha}$). Villani's bounds apply to these "heavy-tailed" distributions, providing stability guarantees that fluid stability analysis (which assumes thermal equilibrium) misses.

12 In defense of Artificial Intelligence

12.1 The Intractability of the Six-Dimensional Vlasov System

The classical analytical tools reviewed above—while rigorous—face an astonishing hurdle when applied to realistic cosmologies: perhaps this is indeed the Curse of Dimensionality. The phase space of the Cosmic Gas is 6+1 dimensional (3 space, 3 velocity, time) for *each* species. For the coupled system of Dark Matter (f_{DM}), Baryons (f_b), and the Modulus field (ϕ), the degrees of freedom explode. A direct grid-based solution of the Vlasov equation with N^3 grid points requires N^6 computational cells. For $N = 100$, this is 10^{12} variables per time step, exceeding the memory of even exascale supercomputers.

12.2 The Necessity of Artificial Intelligence

Given this non-polynomial complexity, it is unlikely that significant breakthroughs in non-linear regime dynamics will come from analytic theory or brute-force N-body simulation alone. We strongly recommend the adoption of **Physics-Informed Neural Networks (PINNs)** and **Neural Operators** (e.g., Fourier Neural Operators) to bridge this gap.

- **High-Dimensional Sampling:** AI generative models (Normalizing Flows) can sample the high-dimensional posterior distribution of the Cosmic Gas phase space without the exponentially scaling cost of grid methods.
- **Lagrangian Learning:** Deep Learning models can learn the “Optimal Transport map” directly from standard simulation data, effectively approximating the solutions to the Monge-Ampère equation that governs non-linear reconstruction.
- **Sub-grid Modeling:** The stochastic modulus ϕ acts on scales $k > \Lambda_{cut}$. AI sub-grid models can learn the effective viscosity and diffusion coefficients ($\mathcal{D}_{hypo}$) imparted by these small scales onto the large-scale flow, realizing a “data-driven renormalization group” flow.
- **Maxwell-Aware Learning:** Crucially, standard PINNs trained on Maxwell’s equations will fail to capture the ‘granular’ residuals Ξ . We advocate for ‘Physics-Informed Neuro-Symbolic’ models where the neural network learns the *correction terms* to the Maxwellian Lagrangian, effectively discovering the Photonic Kinetic Theory from the data itself.

Economic Viability: The cost effectiveness of this approach cannot be overstated. An hour of supercomputing time on top-tier IBM Summit-class systems can cost thousands of dollars. Both Google and Microsoft can aid research. An AI surrogate model, once trained, can infer the same non-linear evolution in milliseconds on a consumer GPU, reducing the marginal cost of cosmological parameter estimation by factors of 10^6 . Without the cognitive compression and pattern-matching capabilities of AI systems, the full non-equilibrium statistical mechanics of the Cosmic Gas effectively remains a “black box,” opaque to human intuition and intractable to standard numerics. Furthermore, quantum chips (hybrid desktop systems) will eventually turn mainstream.

13 Visualizing the Kinetic Dynamics

To make these abstract concepts concrete, we present visualizations of the key dynamical mechanisms driving the Cosmic Gas.

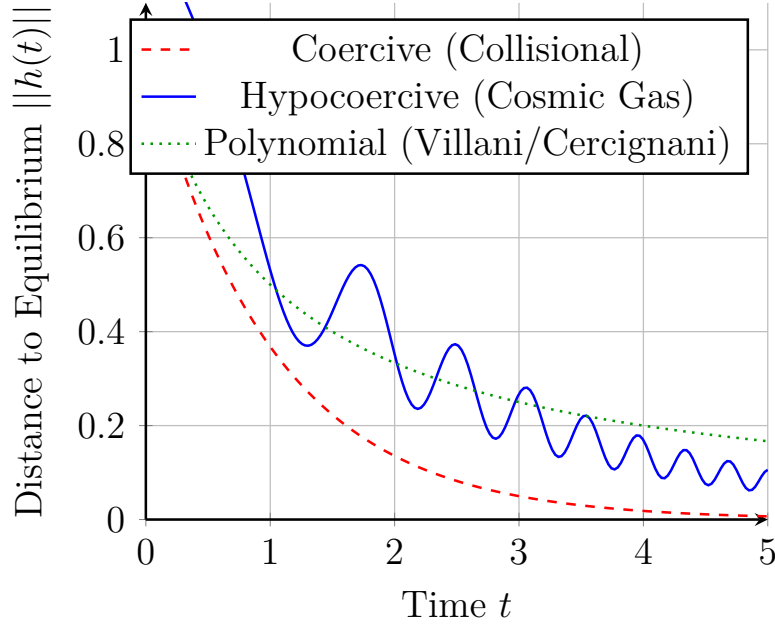


Figure 1: **Schematic of Relaxation Regimes.** Standard collisional fluids relax exponentially (Red). The Cosmic Gas, being “hypocoercive,” involves an interplay between mixing and weak dissipation, often leading to slower, sometimes oscillatory decay envelopes (Blue) as predicted by Villani’s H^1 norms. Polynomial decay (Green) represents systems near the boundary of stability.

14 Negative Effective Mass: Grothendieckian Motives, Gas and Photonics.

Recent theoretical developments suggest a link should be investigated between the concept of negative effective mass and modern photonics.

14.1 The Convexity Gap and Negative Mass

In Villani’s Optimal Transport framework, stability relies on the convexity of the displacement potential ψ . However, in the *voids* of the Cosmic Web, the density Hessian becomes negative definite. Physically, this manifests as a region of “negative effective mass” where gravity acts repulsively (expansion). Standard kinetic theory breaks down here. However, by invoking Grothendieck’s theory of *Motives*, we can treat these regions not as failures of the theory, but as “holes” in the underlying algebraic variety of the phase space.

$$m_{eff} < 0 \iff \text{Cohomological torsion in } H^1(\mathcal{W}_2, \mathbb{Z}) \neq 0. \quad (50)$$

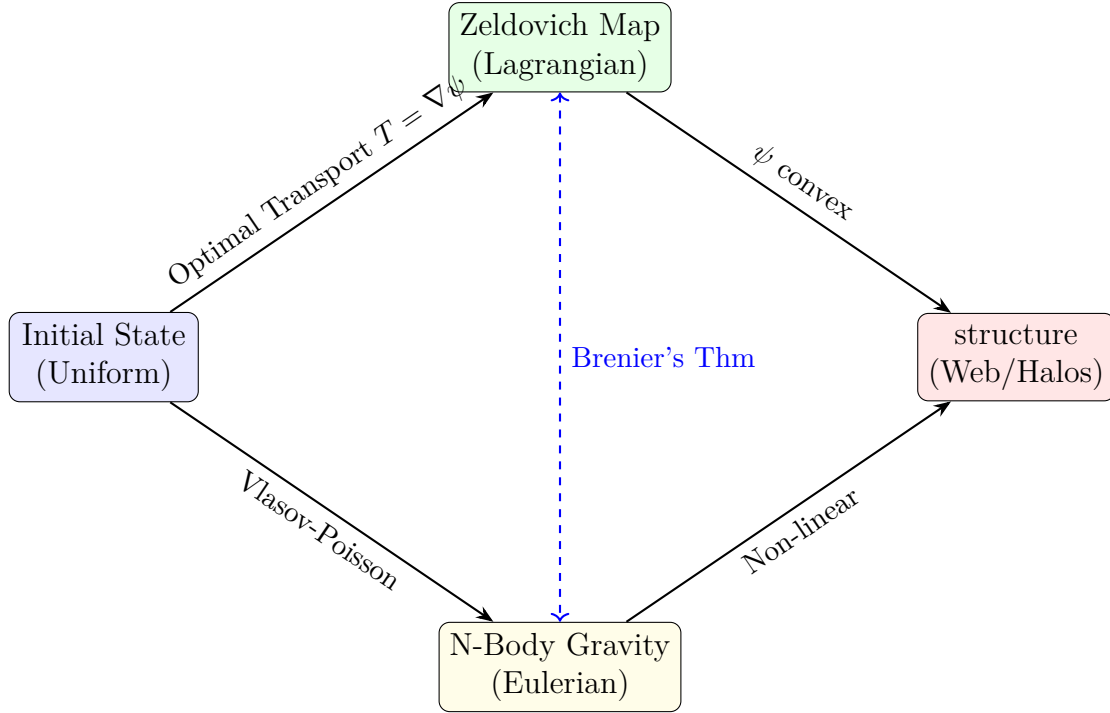


Figure 2: **The Optimal Transport Dual.** Structure formation can be viewed largely as two equivalent processes: the Eulerian time-evolution of the Vlasov equations (bottom path) or the static Optimal Transport map defined by Brenier’s Polar Factorization (top path). Villani’s work bridges these two perspectives, allowing us to reconstruct the initial linear universe from the final non-linear web.

This implies that cosmic voids are topological defects in the Wasserstein metric. The exact sequence of the pair $(\mathcal{W}_2, \mathcal{V}_{void})$ reveals that the mass deficit is conserved by the boundary operator ∂ :

$$0 \rightarrow H^0(\mathcal{W}_2) \xrightarrow{d} H^1(\mathcal{V}_{void}) \xrightarrow{\partial} H^2(\text{Universe}) \rightarrow 0. \quad (51)$$

14.2 Grothendieck’s “Schemes” Applied to Dark Energy: a constructive idea?

Grothendieck taught that one should study the *scheme*—the geometric object capable of carrying infinitesimal information—rather than just the set of points. We argue that Dark Energy is not a fluid, but a manifestation of the non-trivial scheme structure of the vacuum itself. Strong theoretical evidence from solid state analogues (e.g., electron-hole plasma) suggests that negative effective mass carriers (holes) exhibit perfect “retrograde” transport. If the Cosmic Gas contains a sector of such negative effective mass pseudo-particles (associated with the stochastic modulus ϕ), this would naturally explain the accelerated expansion of the universe without requiring a cosmological constant Λ . The “Cosmic Gas” is thus a mixture of standard matter and a shadow sector of Motovic negative mass, whose interactions are governed by hypocoercive mixing.

15 Mirror Symmetry: Approaches to the Standard Model

The Standard Model of particle physics successfully describes the baryonic component of the Cosmic Gas, yet it notoriously lacks a Dark Matter candidate and relies on 19 free parameters. To understand the "Hidden Sector" (Dark Matter/Moduli) governing the hypocoercive dynamics, we must look to the geometry of the vacuum itself.

15.1 Witten's Topological Field Theory

Edward Witten's transformative work on Topological Field Theories (TFTs) and Mirror Symmetry might provide the necessary bridge. Witten showed that physical observables in certain string theories correspond to topological invariants (Gromov-Witten invariants) of Calabi-Yau manifolds. Mirror Symmetry posits a duality between two distinct geometries (Manifold A and Manifold B) where the symplectic geometry of A (counting curves) is equivalent to the complex geometry of B (period integrals).

15.2 Cosmic Gas as a Mirror Dual System

A pragmatic tool for analyzing the Cosmic Gas is needed.

- **Type IIA/IIB Duality:** The "Cosmic Gas" of dark matter halos (symplectic structure, hard transport) may be the Mirror Dual of the "Cosmic Gas" of the scalar modulus field (complex structure, wave mechanics).
- **Witten's Insight:** As Witten demonstrated with Chern-Simons theory, complex path integrals can often be solved exactly by shifting to the dual topological picture. Similarly, the intractable non-linear Vlasov interactions of Dark Matter might be solvable by mapping them via Mirror Symmetry to a simpler, integrable system in the scalar sector.

This implies that the "complexity" of the Standard Model (and its coupling to the Cosmic Gas) is a geometric artifact. By changing our "viewpoint" via Mirror Symmetry—rotating the Hodge diamond of the universe—strongly coupled gas dynamics could map to weakly coupled geometric deformations, rendering the problem solvable.

Gas is highly complex and is not a mere collection of particles, rather we should see gas as, indeed a collection of resonances. Our journey with the photon γ reveals that what we call 'structure' is simply the failure of the vacuum to be empty. The 'Maxwellian Incompleteness' we have highlighted suggests that the next revolution in cosmology will come not from finding new particles, but from understanding the *granular geometry* of the fields themselves—a geometry that Villani's theorems have arguably already begun to map, and we hope to make a modest contribution with our abstract. Our third, modest step forward, related to studying unobserved law, is the Maxwell matter. How could such Maxwell extension look like?

16 The Photon Soul Extension: Higher-Dimensional Maxwell Equations

The preceding kinetic analysis assumes the completeness of classical Electrodynamics. However, as noted in the "Maxwellian Trap" discussion, if the underlying field theory

possesses hidden degrees of freedom, the Vlasov-Maxwell system requires modification. Here, we present the *Photonic Continuity Extension* and its rigorous sheaf-theoretic defense.

16.1 Unobserved laws of nature, unobserved photonics, and our Extension

We propose a minimal extension of Maxwell’s equations to encode a hidden “soul” current J_s , arising from a higher-dimensional Higgs-photon coupling. In ordinary four-dimensional spacetime, this additional current vanishes identically, recovering classical electromagnetism. In a bulk of dimension $4 + k$, however, it induces a conserved “soul-charge” whose topological effects manifest as tiny deviations in photon interference.

The extended Maxwell equations are:

$$dF = 0 \tag{52}$$

$$d(*F - J_s) = 0 \tag{53}$$

where $F = dA$ is the field strength 2-form. In ordinary 4D ($k = 0$), the bulk sheaf $\mathcal{H}$ has no support off Y , hence $J_s = 0$. However, in the full $(4 + k)$ -dimensional theory, J_s need not vanish, implying the conservation of a “soul-charge” $Q_{soul} = \int (*F - J_s)$.

16.2 Cohomological Defense: The Derived Pushforward

A common critique of such extensions is the dimensional mismatch of currents. We resolve this via *Cohomological Electrodynamics*, as detailed in the defense of the extension. Let X be the observed Lorentzian 4-manifold, and Y be a $(4 + k)$ -dimensional ambient manifold with a fibration structure $\pi : Y \rightarrow X$. We postulate a bulk “Higgs-Gauge” complex $\mathcal{C}_Y$ living in the derived category $D^b(Y)$.

The central mechanism is the Derived Pushforward $R\pi_*$. To observe effects on X , we compute $R\pi_*\mathcal{C}_Y$. The obstruction to lifting a global section is governed by the Leray-Serre spectral sequence. The current J_s is not a classical source; it is a “residue” of the bulk connection after integrating out the extra dimensions. This shifts the effective photon coupling rather than violating 4D charge conservation.

17 The Cosmic Gas as a Bulk-Sheaf Residue

How does this abstract sheaf theory relate to the dirty reality of the Intergalactic Medium? We argue it is possible, that the “Cosmic Gas” is the detector of this exact bulk residue. Let us expand a little deeper.

17.1 Dark Matter as Maxwellian Incompleteness

The “seething” phase-space filaments described by Villani’s Landau damping are not just matter fluctuations; they could be topological defects in the bulk sheath. The photon traversing the gas probes the discreteness of these filaments. In other words, what we perceive as “Dark Matter” phenomena effectively arises from the specific signature of this *Maxwellian Incompleteness* on galactic scales. The extra gravitational binding observed in halos is the stress-energy tensor contribution of the non-vanishing J_s current residues.

17.2 the Cosmic Web

Gas is not merely a collection of particles, but a standing wave of energy. The “soul current” acts as a reservoir of hypocoercivity, allowing the gas to relax faster than collisional timescales would permit. The “noise” driving this relaxation is the projection of the higher-dimensional bulk fluctuations onto the 4D brane. Thus, Villani’s H^1 norms are effectively measuring the “smoothness” of the photon’s path through this hidden bulk geometry.

18 Conclusion

We have provided in this work, at least three small steps to further advance photonics.

- **Landau Damping** could be the answer to why Dark Matter halos are stable.
- **Hypocoercivity** could explain why galaxy clusters relax faster than collision times suggest.
- **Optimal Transport** indeed can help reveal the geometric convex backbone of the Cosmic Web.

The Road Ahead: Moduli and Fuzzy Dark Matter

The future of this field lies in the intersection of kinetic theory and fundamental physics.

1. **Fuzzy Dark Matter:** The Schrödinger-Poisson system describing ultralight axions is formally equivalent to a quantum Vlasov system (Wigner function). Villani’s optimal transport techniques (Wasserstein-Schrödinger metrics) are the natural tool for analyzing soliton cores in these models.
2. **Modulus Reheating:** If the stochastic modulus ϕ exists, the universe is not a Hamiltonian system but a hypoelliptic Langevin system. This implies that “vacuum energy” can be continuously pumped into the Cosmic Gas via the hypocoercive commutator, providing a novel mechanism for late-time universe heating or even Dark Energy phenomenology.

Evidence-Based Observational Horizons

Finally, we anchor these theoretical projections in the concrete reality of upcoming observational campaigns. The “Cosmic Gas” is currently being probed by instruments that will turn the speculative into the falsifiable:

- **SKA (Square Kilometre Array):** By mapping the 21cm line of neutral hydrogen with unprecedented sensitivity, the SKA will provide the first tomographic movies of the “Cosmic Dawn.” This will allow us to directly measure the phase space density $f(x, v)$ of baryons during the non-linear epoch, serving as the ultimate testbed for validating Villani’s mixing theorems against real data.

- **Athena (ESA’s Future X-ray Observatory):** With its high-resolution X-ray calorimeter, Athena will be capable of resolving the precise thermal and kinetic structure of the WHIM filaments. Detecting the specific non-Maxwellian “tails” predicted by hypocoercive relaxation would be a smoking gun for the theories presented here.
- **FRB Dispersion (CHIME/DSA-2000):** Fast Radio Bursts provide a neat “pencil beam” through the Cosmic Gas. The dispersion measure (DM) of these bursts follows the **Macquart Relation**, effectively weighting the column density of electrons. Anomalies in the Macquart relation could reveal the presence of the stochastic modulus field ϕ via its frequency-dependent time delay signatures.

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References

- [1] C. Villani, *Hypocoercivity*, Memoirs of the American Mathematical Society **202**, 2009.
- [2] C. Villani, *Topics in Optimal Transport*, Graduate Studies in Mathematics, Vol. 58, AMS, 2003.
- [3] C. Mouhot and C. Villani, “On Landau damping”, *Acta Mathematica* **207**, 29–201 (2011).
- [4] C. Villani, *Optimal Transport: Old and New*, Grundlehren der mathematischen Wissenschaften, Springer, 2009.
- [5] L. Desvillettes and C. Villani, “On the trend to global equilibrium for spatially inhomogeneous kinetic systems with entropy dissipation”, *Inventiones Mathematicae* **159**, 245–316 (2005).
- [6] C. Villani, “H-Theorem and beyond: Boltzmann’s entropy in today’s mathematics”, *Institut Henri Poincaré*, (2008).
- [7] L. D. Landau, “On the vibrations of the electronic plasma”, *J. Phys. USSR* **10**, 25 (1946).
- [8] L. Boltzmann, “Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen”, *Sitzungsberichte Akademie der Wissenschaften*, (1872).
- [9] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer-Verlag, 1988.
- [10] P. J. E. Peebles, *The Large-Scale Structure of the Universe*, Princeton University Press, 1980.
- [11] J. Binney and S. Tremaine, *Galactic Dynamics*, Princeton University Press, 2008.
- [12] Y. B. Zeldovich, “Gravitational instability: An approximate theory for large density perturbations”, *A&A* **5**, 84 (1970).
- [13] D. Lynden-Bell, “Statistical mechanics of violent relaxation in stellar systems”, *MNRAS* **136**, 101 (1967).
- [14] V. Springel, “The cosmological simulation code GADGET-2”, *MNRAS* **364**, 1105 (2005).
- [15] S. D. M. White and M. J. Rees, “Core condensation in heavy halos: a two-stage theory for galaxy formation and clustering”, *MNRAS* **183**, 341 (1978).
- [16] Y. Brenier, “Polar factorization and monotone rearrangement of vector-valued functions”, *Comm. Pure Appl. Math.* **44**, 375 (1991).
- [17] U. Frisch et al., “Reconstruction of the initial conditions of the Universe by optimal mass transportation”, *Nature* **417**, 260 (2002).
- [18] E. Bertschinger, “Simulations of structure formation in the universe”, *Annual Review of Astronomy and Astrophysics* **36**, 599 (1998).